

Introduction to Calculus of Variations

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Euler-Lagrange Equation

1.1 Functionals

The calculus of variations deals with minimization or maximization of functions of functions, or *functionals*. A functional is a mapping or a transformation that depends on one or more functions and the values of functionals are numbers.

Example 1.1.1 Suppose $x(t)$ is a continuous function of t defined in the interval $[t_0, t_f]$ and:

$$J(x) = \int_{t_0}^{t_f} x(t) dt \quad (1.1)$$

The real number assigned by the functional J is the area under the $x(t)$ curve.

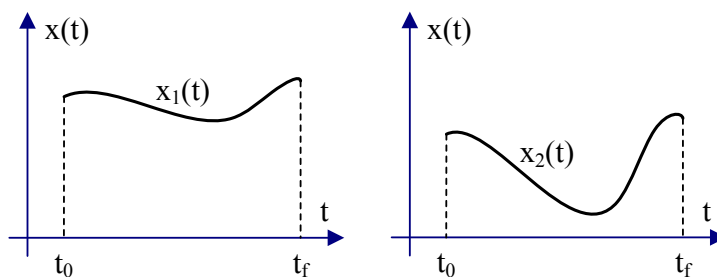


Figure 1.1: Area under a curve

As shown in Figure 1.1, for various functions $x(t)$, the functional has different real values.

Example 1.1.2 The length l , of an arc connecting two given points $A(x_0, y_0)$, $B(x_1, y_1)$ in the $x-y$ plane as depicted in Figure 1.2 is an example of a functional. We know that the length can be calculated as, (***, 2001):

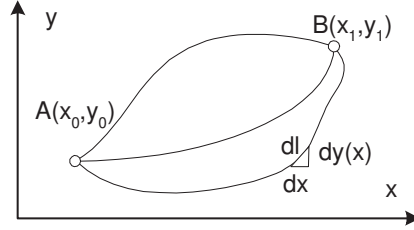


Figure 1.2: An arc connecting two points

$$l = l(y(x)) = \int_{x_0}^{x_1} dl(x) = \int_{x_0}^{x_1} \sqrt{dx^2 + dy(x)^2} = \int_{x_0}^{x_1} \sqrt{1 + \left(\frac{dy(x)}{dx}\right)^2} dx$$

subject to the constraints:

$$y(x_0) = y_0, \quad y(x_1) = y_1 \quad (1.2)$$

Example 1.1.3 The Brachistochrone problem. The problem originally gave rise to the calculus of variations (Johann Bernoulli posed it in 1696). The term derives from the Greek brachistos "the shortest" and chronos "time, delay".

We wish to find the curve that minimizes the time necessary for a bead attached to a wire to go from $(0, y_1)$ to $(x_1, 0)$ under the influence of a constant gravitational field

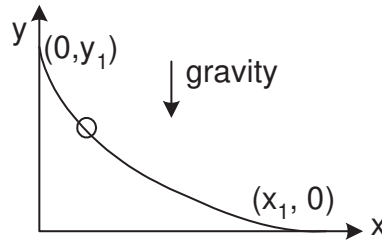


Figure 1.3: Brachistochrone problem

The time to travel from point $(0, y_1)$ to $(x_1, 0)$ is:

$$T = \int_0^T dt \quad (1.3)$$

If the length of the curve is l , and ds is an infinitely small part of the curve, we obtain:

$$T = \int_0^T dt = \int_0^l \frac{ds}{v} \quad (1.4)$$

where v is the speed of the bead. Since $ds = \sqrt{dx^2 + dy^2}$, the equation (1.4) is written as:

$$T = \int_0^{x_1} \frac{\sqrt{dx^2 + dy^2}}{v} = \int_0^{x_1} \frac{1}{v} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_0^{x_1} \frac{1}{v} \sqrt{1 + y'^2} dx \quad (1.5)$$

The speed at any point can be obtain from energy conservation, equating kinetic energy to gravitational potential energy:

$$\frac{1}{2}mv^2 = 2gy, \Rightarrow v = \sqrt{2gy} \quad (1.6)$$

The functional to be minimized then gives:

$$T = \int_0^{x_1} \sqrt{\frac{1 + y'^2}{2gy}} dx = \int_0^{x_1} L(y, y') dx \quad (1.7)$$

where $y = y(x)$.

1.2 Euler-Lagrange Equation

Problem: Suppose we are given two points (t_0, x_0) and (t_f, x_f) in the (t, x) plane. We wish to find a curve, trajectory, joining the given points such that the functional:

$$J = \int_{t_0}^{t_f} g(t, x, \dot{x}) dt$$

along this trajectory can achieve its extremal, that is maximal or minimal value, (Kirk, 2004).

A fundamental equation of calculus of variations states that J has a stationary value if the **Euler-Lagrange equation**:

$$\frac{\partial g}{\partial x} - \frac{d}{dt} \left(\frac{\partial g}{\partial \dot{x}} \right) = 0 \quad (1.8)$$

is satisfied.

The curves $x = x(t)$ that are solutions to the Euler-Lagrange equation are called *extremals*. We shall write the above equation also as:

$$g_x - \frac{d}{dt} g_{\dot{x}} = 0$$

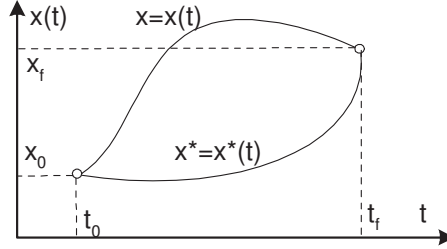


Figure 1.4: Trajectories between two points

Example 1.2.1 Find the extremal for the functional:

$$J = \int_0^{\pi/2} (\dot{x}(t)^2 - x(t)^2) dt = \int_0^{\pi/2} (\dot{x}^2 - x^2) dt$$

where $x(0) = 0$, $x(\pi/2) = 1$. The Euler-Lagrange equation for the problem takes the form:

$$\begin{aligned} g_x - \frac{d}{dt} g_{\dot{x}} &= \frac{\partial}{\partial x} (\dot{x}^2 - x^2) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} (\dot{x}^2 - x^2) \right) = 0 \\ -2x - \frac{d}{dt} (2\dot{x}) &= -2x - 2\ddot{x} = 0 \end{aligned}$$

or

$$\ddot{x} + x = 0$$

Obs. $a\ddot{x} + b\dot{x} + cx = 0$ has the characteristic equation $as^2 + bs + c = 0$. If the roots of the characteristic equation are real: $s_{1,2} \in \mathbb{R}$, the general solution is: $x(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t}$. If the roots of the characteristic equation are complex $s_{1,2} = \alpha \pm j\beta$, the general solution is: $x(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t$

The characteristic equation $s^2 + 1 = 0$ has two complex roots $s_{1,2} = \pm j$ and the solution is:

$$x(t) = c_1 \cos t + c_2 \sin t$$

Taking into account the end-point conditions:

$$\begin{aligned} x(0) &= c_1 \cos 0 + c_2 \sin 0 = c_1 = 0 \\ x(\pi/2) &= c_1 \cos \pi/2 + c_2 \sin \pi/2 = c_2 = 1 \end{aligned}$$

Thus we have the extremal $x(t) = \sin t$. The functional J can be minimized or maximized only on the extremal $x(t) = \sin t$. The extremal value of the functional is:

$$J^*(\sin t) = \int_0^{\pi/2} \cos(t)^2 - \sin(t)^2 dt = \int_0^{\pi/2} \cos(2t) dt = \frac{1}{2} (\sin(\pi) - \sin(0)) = 0$$

Example 1.2.2 Find the extremal for the functional

$$J = \int_0^1 (\dot{x}(t)^2 + 12tx(t))dt$$

where $x(0) = 0$, $x(1) = 1$.

The Euler-Lagrange equation for this functional is:

$$g_x - \frac{d}{dt}g_{\dot{x}} = \frac{\partial}{\partial x}(\dot{x}^2 + 12tx) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}}(\dot{x}^2 + 12tx) \right) = 0$$

$$12t - \frac{d}{dt}(2\dot{x}) = 12t - 2\ddot{x} = 0$$

or

$$\ddot{x}(t) = 6t, \quad \dot{x}(t) = 6\frac{t^2}{2} + c_1$$

$$x(t) = \frac{3t^3}{3} + c_1t + c_2 = t^3 + c_1t + c_2$$

$$x(0) = c_2 = 0, \quad x(1) = 1 + c_1 + c_2 = 1 \Rightarrow c_1 = 0$$

The extremal has the form: $x(t) = t^3$.

1.2.1 Transversality conditions

Final time specified, $x(t_f)$ free

Problem: Find a necessary condition for a function to be extremal for a functional

$$J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)dt$$

where t_0 , $x(t_0)$ and t_f are specified and $x(t_f)$ is free.

The admissible curves all begin at the same point and terminate on a vertical line as for example in the case presented in Figure 1.5. We shall call

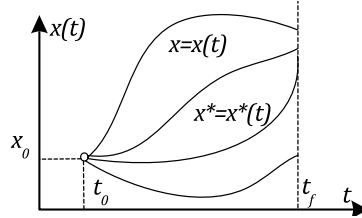


Figure 1.5: Free final state

the necessary condition:

$$\frac{\partial}{\partial \dot{x}} g(x, \dot{x}, t)|_{t=t_f} = 0 \quad (1.9)$$

the natural boundary condition or *the first transversality condition*.

Example 1.2.3 Determine the extremal for the functional

$$J = \int_0^2 [\dot{x}(t)^2 + 2x(t)\dot{x}(t) + 4x(t)^2] dt$$

where $x(0) = 1$, $x(2)$ is free.

The Euler-Lagrange equation:

$$g_x - \frac{d}{dt} g_{\dot{x}} = \frac{\partial}{\partial x} (\dot{x}^2 + 2x\dot{x} + 4x^2) - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} (\dot{x}^2 + 2x\dot{x} + 4x^2) \right) = 0$$

$$2\dot{x} + 8x - \frac{d}{dt} (2\dot{x} + 2x) = 2\dot{x} + 8x - 2\ddot{x} - 2\dot{x} = 0$$

or $\ddot{x} - 4x = 0$.

The characteristic equation is $s^2 - 4 = 0$ with the solutions: $s_{1,2} = \pm 2$.
The general solution is:

$$x(t) = c_1 e^{2t} + c_2 e^{-2t}$$

Evaluating the transversality condition (1.9) with $t = 2$:

$$\frac{\partial g}{\partial \dot{x}}|_{t=2} = (2\dot{x}(t) + 2x(t))_{t=2} = \dot{x}(2) + x(2) = 0$$

where

$$x(t) = c_1 e^{2t} + c_2 e^{-2t}, \quad \dot{x}(t) = 2c_1 e^{2t} - 2c_2 e^{-2t}$$

We obtain:

$$2c_1 e^4 - 2c_2 e^{-4} + c_1 e^4 + c_2 e^{-4} = 0, \quad 3c_1 e^4 - c_2 e^{-4} = 0$$

From $x(0) = 1$:

$$c_1 e^{2 \cdot 0} + c_2 e^{-2 \cdot 0} = 1, \Rightarrow c_1 + c_2 = 1$$

The constants c_1, c_2 result:

$$c_1 = \frac{1}{3e^4 + e^{-4}}, \quad c_2 = \frac{3e^4 + e^{-4} - 1}{3e^4 + e^{-4}}$$

Exercise 1.1 . Determine the smooth curve of smallest length connecting the point $x(0) = 1$ to the line $t = 5$, (Figure 1.6).

Hint. Find the extremal for the functional:

$$J(x) = \int_{t_0}^{t_f} \sqrt{1 + \left(\frac{dx}{dt} \right)^2} dt = \int_0^5 \sqrt{1 + \dot{x}(t)^2} dt$$

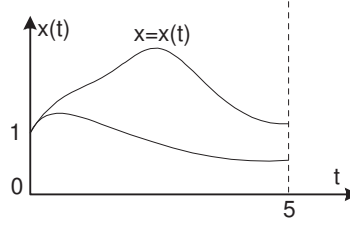


Figure 1.6: The curve connecting a point to a line

Final time free, $x(t_f)$ specified

Problem: Find a necessary condition for a function to be extremal for a functional

$$J = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t) dt$$

where t_0 , $x(t_0) = x_0$ and $x(t_f)$ are specified and t_f is free.

The second *transversality condition* that is the necessary condition needed is:

$$\left(g - \dot{x} \frac{\partial g}{\partial \dot{x}} \right) \Big|_{t=t_f} = 0, \text{ or } (g - \dot{x} g_{\dot{x}}) \Big|_{t=t_f} = 0 \quad (1.10)$$

Example 1.2.4 Find an extremal for the functional:

$$J = \int_1^{t_f} (2x(t) + \frac{1}{2}\dot{x}(t)^2) dt$$

with $x(1) = 4$, $x(t_f) = 4$ and $t_f > 1$ free.

$$g_x - \frac{d}{dt} g_{\dot{x}} = 2 - \frac{d}{dt} \left(\frac{\partial}{\partial \dot{x}} (2x - \frac{1}{2}\dot{x}^2) \right) = 2 - \ddot{x} = 0$$

or

$$\ddot{x} = 2$$

with the solution $\dot{x}(t) = 2t + c_1$, $x(t) = t^2 + c_1 t + c_2$.

The final time t_f is unspecified so, from (1.10):

$$(g - \dot{x} g_{\dot{x}}) \Big|_{t=t_f} = (2x(t) + \frac{1}{2}\dot{x}(t)^2 - \dot{x}(t) \cdot \dot{x}(t)) \Big|_{t=t_f} = 0$$

$$2x(t_f) - \frac{1}{2}\dot{x}(t_f)^2 = 0 \Rightarrow 2(t_f^2 + c_1 t_f + c_2) - \frac{1}{2}(2t_f + c_1)^2 = 0$$

From

$$4t_f^2 + 4c_1t_f + 4c_2 - 2t_f - c_1 = 0$$

together with the boundary conditions:

$$x(1) = 1 + c_1 + c_2 = 4, \quad x(t_f) = t_f^2 + c_1t_f + c_2 = 4$$

the constants and the final time can be calculated and the extremal result as:

$$x(t) = t^2 - 6t + 9, \quad \text{and} \quad t_f = 5$$

1.2.2 Functionals involving several independent variables

In the $n+1$ dimensional space: $(t, x_1, x_2, \dots, x_n)$ the functional

$$J = \int_{t_0}^{t_f} g(t, x_1, x_2, \dots, x_n, \dot{x}_1, \dot{x}_2, \dots, \dot{x}_n) dt$$

can only be extremized on the trajectories that satisfy the n Euler-Lagrange equations:

$$\frac{\partial g}{\partial x_i} - \frac{d}{dt} \left(\frac{\partial g}{\partial \dot{x}_i} \right) = 0, \quad \text{or} \quad g_{x_i} - \frac{d}{dt} g_{\dot{x}_i} = 0, \quad i = 1, \dots, n$$

Using matrix-vector notation the functional, the Euler-Lagrange equation and the transversality conditions are stated as:

$$J = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) dt \tag{1.11}$$

$$\frac{\partial g}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial g}{\partial \dot{\mathbf{x}}} \right) = 0 \tag{1.12}$$

$$\frac{\partial g}{\partial \dot{\mathbf{x}}} \Big|_{t=t_f} = 0 \tag{1.13}$$

$$\left(g - \left[\frac{\partial g}{\partial \dot{\mathbf{x}}} \right]^T \dot{\mathbf{x}} \right) \Big|_{t=t_f} = 0 \tag{1.14}$$

where:

$$\mathbf{x} = \mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \dots \\ x_n(t) \end{bmatrix}, \quad \dot{\mathbf{x}} = \dot{\mathbf{x}}(t) = \begin{bmatrix} \frac{d}{dt} x_1(t) \\ \dots \\ \frac{d}{dt} x_n(t) \end{bmatrix}$$

Example 1.2.5 Find an extremal for the functional

$$J(\mathbf{x}) = \int_0^{\pi/4} [x_1^2(t) + \dot{x}_1(t)\dot{x}_2(t) + \dot{x}_2^2(t)]dt$$

The functions x_1 and x_2 are independent and the boundary conditions are:

$$x_1(0) = 1; \quad x_1(\pi/4) = 2; \quad x_2(0) = 3/2; \quad x_2(\pi/4) \text{ free}$$

The Euler-Lagrange equations are:

$$g_{x_1} - \frac{d}{dt}g_{\dot{x}_1} = 2x_1 - \frac{d}{dt}(\dot{x}_2) = 2x_1 - \ddot{x}_2 = 0 \quad (1.15)$$

$$g_{x_2} - \frac{d}{dt}g_{\dot{x}_2} = -\frac{d}{dt}(\dot{x}_1 + 2\dot{x}_2) = -\ddot{x}_1 - 2\ddot{x}_2 = 0 \quad (1.16)$$

From (1.15): $\ddot{x}_2(t) = 2x_1(t)$. By replacing in (1.16) we obtain:

$$\ddot{x}_1 + 4x_1 = 0$$

which has the solution:

$$x_1(t) = c_1 \cos 2t + c_2 \sin 2t$$

therefore

$$\ddot{x}_2(t) = 2c_1 \cos 2t + 2c_2 \sin 2t$$

Integrating twice yields:

$$x_2(t) = -\frac{c_1}{2} \cos 2t - \frac{c_2}{2} \sin 2t + c_3 t + c_4$$

From the first transversality condition (t_f is specified and $x_2(t_f)$ is free:

$$\left(\frac{\partial g}{\partial \dot{x}_2} \right)_{t=\pi/4} = 0$$

$$g_{\dot{x}_2} = \dot{x}_1 + 2\dot{x}_2 = 2c_3 = 0, \Rightarrow c_3 = 0$$

From the specified boundary conditions we have:

$$x_1(0) = c_1 \cdot 1 + c_2 \cdot 0 = 1 \Rightarrow c_1 = 1$$

$$x_2(0) = -\frac{c_1}{2} \cdot 1 - \frac{c_2}{2} \cdot 0 + c_3 \cdot 0 + c_4 = \frac{3}{2}, \Rightarrow c_4 = 1.5 + \frac{c_1}{2} = 2$$

$$x_1(\pi/4) = c_1 \cdot 0 + c_2 \cdot 1 = 2, \Rightarrow c_2 = 2$$

The extremals are then:

$$x_1(t) = \cos 2t + 2 \sin 2t, \quad x_2(t) = -\frac{1}{2} \cos 2t - \sin 2t + 2$$

1.3 Constrained minimization of functionals

1.3.1 Point constraints

Let us determine a set of necessary conditions for a function $\mathbf{x}^*(t)$ to be an extremal for a functional of the form:

$$J = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) dt \quad (1.17)$$

where $\mathbf{x}(t)$ is a $n \times 1$ vector of functions, that is required to satisfy m relationships of the form:

$$f_i(\mathbf{x}(t), t) = 0, \quad i = 1, 2, \dots, m < n \quad (1.18)$$

which are called point constraints. Constraints of this type would be present if, for example, the admissible trajectories were required to lie on a specified surface in the $n + 1$ dimensional $(\mathbf{x}(t), t)$ space. The presence of these m constraining relations means that only $n - m$ of the n components of $\mathbf{x}$ are independent.

One way to attack this problem might be to solve equations (1.18) for m of the components of $\mathbf{x}(t)$ in terms of the remaining $n - m$ components - which can be regarded as $n - m$ independent functions - and use these equations to eliminate the m dependent components of $\mathbf{x}(t)$ and $\dot{\mathbf{x}}(t)$ from J .

Unfortunately, the constraining equations are generally nonlinear algebraic equations, which may be quite difficult to solve.

As an alternative approach we can use Lagrange multipliers. The first step is to form an *augmented functional* by adjoining the constraining relations to J , which yields:

$$\begin{aligned} J_a(\mathbf{x}, \lambda) &= \int_{t_0}^{t_f} [g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \lambda_1(t)f_1(\mathbf{x}(t), t) + \dots + \lambda_m(t)f_m(\mathbf{x}(t), t)] dt \\ &= \int_{t_0}^{t_f} [g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \lambda^T(t)\mathbf{f}(\mathbf{x}(t), t)] dt \end{aligned} \quad (1.19)$$

Since the constraints must be satisfied for all $t \in [t_0, t_f]$, the Lagrange multipliers $\lambda_1, \dots, \lambda_m$ are assumed to be functions of time. This allows us the flexibility of multiplying the constraining relations by a different real number for each value of t .

If we define the *augmented integrand function* as:

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \lambda(t), t) = g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \lambda^T(t)\mathbf{f}(\mathbf{x}(t), t)$$

The Euler-Lagrange equation is written:

$$\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}, \dot{\mathbf{x}}, \lambda, t) - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}, \dot{\mathbf{x}}, \lambda, t) \right) = 0 \quad (1.20)$$

Equations (1.20) are a set of n second-order differential equations, and the constraining relations (1.18) are a set of m algebraic equations. Together these $n + m$ equations constitute a set of necessary conditions for $\mathbf{x}(t)$ to be an extremal.

Example 1.3.1 Find necessary conditions that must be satisfied by the curve of smallest length which lies on the sphere $x_1^2(t) + x_2^2(t) + t^2 = R^2$, for $t \in [t_0, t_f]$, and joins the specified points $\mathbf{x}_0, t_0$ and $\mathbf{x}_f, t_f$.

The functional to be minimized is:

$$J(\mathbf{x}) = \int_{t_0}^{t_f} \sqrt{1 + \dot{x}_1^2(t) + \dot{x}_2^2(t)} dt$$

The augmented integrand function is:

$$g_a(\mathbf{x}, \dot{\mathbf{x}}, \lambda, t) = \sqrt{1 + \dot{x}_1^2(t) + \dot{x}_2^2(t)} + \lambda(t)(x_1^2(t) + x_2^2(t) + t^2 - R^2)$$

Performing the operations indicated by (1.20) gives:

$$\begin{aligned} \frac{\partial g_a}{\partial x_1} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{x}_1} \right) &= 2x_1\lambda - \frac{d}{dt} [\dot{x}_1(1 + \dot{x}_1^2 + \dot{x}_2^2)^{-1/2}] = 0 \\ \frac{\partial g_a}{\partial x_2} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{x}_2} \right) &= 2x_2\lambda - \frac{d}{dt} [\dot{x}_2(1 + \dot{x}_1^2 + \dot{x}_2^2)^{-1/2}] = 0 \end{aligned}$$

In addition, of course, it is necessary that the constraint relation:

$$x_1^2(t) + x_2^2(t) + t^2 = R^2$$

is satisfied.

1.3.2 Differential equation constraints

Let us now find necessary conditions for a function $\mathbf{x}(t)$ to be an extremal for the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) dt$$

$\mathbf{x}$ is a $(n + m) \times 1$ vector of functions which must satisfy the n differential equations:

$$f_i(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0, \quad i = 1, 2, \dots, n$$

Because of the n differential equations constraints, only m components of $\mathbf{x}$ are independent. Constraints of this type may represent the state equation constraints in optimal control problems where $\mathbf{x}$ corresponds to the $n + m$ vector $[\mathbf{x}|\mathbf{u}]^T$.

As with point constraints, it is generally not feasible to eliminate n dependent functions and their derivatives from the functional J so we shall use the method of Lagrange multipliers.

If we define the augmented integrand function as:

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), \lambda(t), t) = g(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) + \lambda(t)^T \mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t)$$

where $\lambda(t)$ is a vector of n functions of time (Lagrange multipliers).

The Euler- Lagrange equation can be written:

$$\frac{\partial g_a}{\partial \mathbf{x}}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) \right] = 0 \quad (1.21)$$

and the constraints can be written in the vector form:

$$\mathbf{f}(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = 0 \quad (1.22)$$

Equations (1.21) and (1.22) compose a set of $2n + m$ second-order differential equations.

Example 1.3.2 Find the equations that must be satisfied by an extremal for the functional

$$J(\mathbf{x}) = \int_{t_0}^{t_f} \frac{1}{2} [x_1^2(t) + x_2^2(t)] dt$$

where $x_1(t)$ and $x_2(t)$ are related by:

$$\dot{x}_1(t) = x_2(t)$$

There is one constraint, so the function f in (1.22) is:

$$f(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = x_2(t) - \dot{x}_1(t)$$

and one Lagrange multiplier is required.

The augmented function g_a is:

$$g_a(\mathbf{x}(t), \dot{\mathbf{x}}(t), t) = \frac{1}{2} [x_1^2(t) + x_2^2(t)] + \lambda(x_2(t) - \dot{x}_1(t))$$

From equation (1.21) we have:

$$\frac{\partial g_a}{\partial x_1} - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{x}_1} \right] = x_1(t) - \frac{d}{dt}(\lambda(t)) = x_1(t) + \dot{\lambda}(t) = 0 \quad (1.23)$$

$$\frac{\partial g_a}{\partial x_2} - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{x}_2} \right] = x_2(t) + \lambda(t) = 0 \quad (1.24)$$

We need also:

$$\dot{x}_1(t) = x_2(t) \quad (1.25)$$

Equations (1.23), (1.24), (1.25) are necessary conditions for $\mathbf{x}^*(t) = [x_1(t) \ x_2(t)]^T$ to be an extremal for the functional J .

The solution of system (1.23, 1.24, 1.25) can be found as follows: from (1.24) we have $\lambda(t) = -x_2(t)$ which, if differentiated with respect to t and replaced in (1.23) gives: $\dot{x}_2(t) = x_1(t)$, or (using (1.25)) $\ddot{x}_2(t) = x_2(t)$.

The solution result as:

$$x_1(t) = C_1 \cdot e^t - C_2 \cdot e^{-t}, \quad x_2(t) = C_1 \cdot e^t + C_2 \cdot e^{-t} \quad (1.26)$$

For a set of initial conditions $x_1(0) = x_{10}$, $x_2(0) = x_{20}$, the constants C_1 and C_2 can be calculated and the solution result as:

$$x_1(t) = \frac{x_{10} + x_{20}}{2} \cdot e^t - \frac{x_{20} - x_{10}}{2} \cdot e^{-t}, \quad x_2(t) = \frac{x_{10} + x_{20}}{2} \cdot e^t + \frac{x_{20} - x_{10}}{2} \cdot e^{-t} \quad (1.27)$$

Example 1.3.3 Suppose that the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_1(t) \\ \dot{x}_2(t) &= -2x_1(t) - 3x_2(t) + u(t) \end{aligned}$$

is to be controlled to minimize the performance measure

$$J(\mathbf{x}, \mathbf{u}) = \int_{t_0}^{t_f} \frac{1}{2} [x_1^2(t) + x_2^2(t) + u^2(t)] dt$$

Find a set of necessary conditions for optimal control.

The differential equation constraints are then:

$$\begin{aligned} f_1(\mathbf{x}, u) &= x_2(t) - x_1(t) - \dot{x}_1(t) \\ f_2(\mathbf{x}, u) &= -2x_1(t) - 3x_2(t) + u(t) - \dot{x}_2(t) \end{aligned}$$

The augmented function g_a :

$$\begin{aligned} g_a(\mathbf{x}, u, \lambda) &= \frac{1}{2} [x_1^2(t) + x_2^2(t) + u^2(t)] + \lambda_1(x_2(t) - x_1(t) - \dot{x}_1(t)) + \\ &\quad + \lambda_2(-2x_1(t) - 3x_2(t) + u(t) - \dot{x}_2(t)) \end{aligned}$$

The Euler equations:

$$\begin{aligned} \frac{\partial g_a}{\partial x_1} - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{x}_1} \right] &= x_1 - \lambda_1 - 2\lambda_2 + \dot{\lambda}_1 = 0 \\ \frac{\partial g_a}{\partial x_2} - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{x}_2} \right] &= x_2 + \lambda_1 - 3\lambda_2 + \dot{\lambda}_2 = 0 \\ \frac{\partial g_a}{\partial u} - \frac{d}{dt} \left[\frac{\partial g_a}{\partial \dot{u}} \right] &= u + \lambda_2 = 0 \end{aligned}$$

We obtain the differential equations:

$$\begin{aligned}\dot{\lambda}_1 &= -x_1 + \lambda_1 + 2\lambda_2 \\ \dot{\lambda}_2 &= -x_2 - \lambda_1 + 3\lambda_2\end{aligned}$$

and the algebraic equation (since u does not appear in g_a)

$$u + \lambda_2 = 0$$

The two additional equations that must be satisfied by an extremal are the constraints:

$$\begin{aligned}\dot{x}_1 &= x_2 - x_1 \\ \dot{x}_2 &= -2x_1 - 3x_2 + u\end{aligned}$$

Variational approach for optimal control

2.1 Necessary conditions for optimal control

In this section we shall apply variational methods to optimal control problems.

We consider the problem of minimizing the performance measure

$$J = h(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} g(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (2.1)$$

subject to:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad \mathbf{x}(t_0) = \mathbf{x}_0 \quad (2.2)$$

The problem is to find an admissible control $\mathbf{u}^*(t)$ that causes the system (2.2) to follow an admissible trajectory $\mathbf{x}^*(t)$ that minimizes the performance measure (2.1).

We shall initially assume that the admissible state and control are not bounded and that the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ and the initial time t_0 are specified. $\mathbf{x}(t)$ is the $n \times 1$ state vector and $\mathbf{u}(t)$ is the $m \times 1$ vector of control inputs.

In the terminology of the previous section (Euler-Lagrange equation with differential equation constraints) we have a problem involving $n+m$ functions which must satisfy the n differential equation constraints. The m control inputs are the independent functions. The only difference between (2.1) and the functionals considered in the previous section is the term involving the final states and final time. Assuming that h is a differentiable function we can write:

$$h(\mathbf{x}(t_f), t_f) = \int_{t_0}^{t_f} \frac{d}{dt} [h(\mathbf{x}(t), t)] dt + h(\mathbf{x}(t_0), t_0)$$

so that the performance measure can be expressed as:

$$J = \int_{t_0}^{t_f} \left[g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}(h(\mathbf{x}(t), t)) \right] dt + h(\mathbf{x}(t_0), t_0)$$

Since $\mathbf{x}_0$ and t_0 are fixed, the minimization does not affect the $h(\mathbf{x}(t_0), t_0)$ term, so we need consider only the functional:

$$J(\mathbf{x}, \mathbf{u}) = \int_{t_0}^{t_f} \left[g(\mathbf{x}(t), \mathbf{u}(t), t) + \frac{d}{dt}(h(\mathbf{x}(t), t)) \right] dt$$

Using the chain rule of differentiation, we find that this becomes:

$$J(\mathbf{x}, \mathbf{u}) = \int_{t_0}^{t_f} \left\{ g(\mathbf{x}(t), \mathbf{u}(t), t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t), t) \right]^T \dot{\mathbf{x}}(t) + \frac{\partial h}{\partial t}(\mathbf{x}(t), t) \right\} dt$$

Obs. The derivative of $h(\mathbf{x}(t), t)$ with respect to time was calculated:

$$\frac{d}{dt}h(\mathbf{x}(t), t) = \frac{\partial h}{\partial \mathbf{x}} \frac{d\mathbf{x}}{dt} + \frac{\partial h}{\partial t} = \frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial h}{\partial t}$$

To include the differential equation constraints we form the augmented function g_a by introducing the Lagrange multipliers $\lambda^T(t) = [\lambda_1(t) \dots \lambda_n(t)]^T$:

$$g_a(\mathbf{x}, \mathbf{u}, \lambda, t) = g(\mathbf{x}, \mathbf{u}, t) + \left[\frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}, t) \right]^T \dot{\mathbf{x}} + \frac{\partial h}{\partial t}(\mathbf{x}, t) + \lambda^T(\mathbf{f}(\mathbf{x}, \mathbf{u}, t) - \dot{\mathbf{x}})$$

The Euler-Lagrange equations are:

$$\begin{aligned} \frac{\partial g_a}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{\mathbf{x}}} \right) &= 0 \\ \frac{\partial g_a}{\partial \mathbf{u}} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{\mathbf{u}}} \right) &= 0 \end{aligned}$$

$$\begin{aligned} \frac{\partial g_a}{\partial \mathbf{x}} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{\mathbf{x}}} \right) &= \frac{\partial}{\partial \mathbf{x}} \left[g + \frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial h}{\partial t} + \lambda^T(f - \dot{\mathbf{x}}) \right] - \frac{d}{dt} \left(\frac{\partial h}{\partial \mathbf{x}} - \lambda \right) = \\ &= \frac{\partial}{\partial \mathbf{x}}(g + \lambda^T f) + \dot{\lambda} + \frac{\partial}{\partial \mathbf{x}} \left(\frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial h}{\partial t} \right) - \frac{d}{dt} \left(\frac{\partial h}{\partial \mathbf{x}} \right) = \\ &= \frac{\partial}{\partial \mathbf{x}}(g + \lambda^T f) + \dot{\lambda} + \frac{\partial^2 h}{\partial \mathbf{x}^2} \dot{\mathbf{x}} + \frac{\partial^2 h}{\partial \mathbf{x} \partial t} - \frac{\partial^2 h}{\partial \mathbf{x}^2} \frac{\partial \mathbf{x}}{\partial t} - \frac{\partial^2 h}{\partial \mathbf{x} \partial t} \\ &= \frac{\partial}{\partial \mathbf{x}}(g + \lambda^T f) + \dot{\lambda} = 0 \end{aligned} \tag{2.3}$$

$$\frac{\partial g_a}{\partial \mathbf{u}} - \frac{d}{dt} \left(\frac{\partial g_a}{\partial \dot{\mathbf{u}}} \right) = \frac{\partial g_a}{\partial \mathbf{u}} = \frac{\partial}{\partial \mathbf{u}} (g + \lambda^T f) \quad (2.4)$$

In the following we shall find it convenient to use the function H called *the hamiltonian*, defined as:

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = g(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t) f(\mathbf{x}(t), \mathbf{u}(t), t) = g + \lambda^T f$$

Using this notation, from (2.3) and (2.4) we can write the necessary conditions as follows:

$$\dot{\lambda} = - \frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) \quad (2.5)$$

$$0 = \frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) \quad (2.6)$$

together with the equations (2.2), which are constraints for the optimal control problem:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

The differential equation (2.5) is called the *costate equation* and $\lambda(t)$ is the *costate vector*.

2.1.1 Problems with fixed final time

If the final time t_f is specified, the final state $\mathbf{x}(t_f)$ may be free or specified.

Case I Final state specified.

$$\mathbf{x}(t_f) = \mathbf{x}_f$$

are n relations which are boundary conditions for differential equations (2.5) and (2.2). Together with the initial conditions $\mathbf{x}(t_0) = \mathbf{x}_0$ will ensure a unique solution of the system of differential equations

Case II Final state free. The final condition can be calculated from the first transversality condition:

$$\left(\frac{\partial g_a}{\partial \dot{\mathbf{x}}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) \right)_{t=t_f} = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t_f), t_f) - \lambda(t_f) = 0$$

Thus:

$$\lambda(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t_f), t_f) \quad (2.7)$$

2.1.2 Problems with free final time

Case I Final state $\mathbf{x}(t_f) = \mathbf{x}_f$ fixed, t_f free. The second transversality condition calculated for g_a is:

$$\left(g_a - \dot{\mathbf{x}} \frac{\partial g_a}{\partial \dot{\mathbf{x}}} \right)_{t=t_f} = 0$$

$$\left[g + \frac{\partial h}{\partial \mathbf{x}} \dot{\mathbf{x}} + \frac{\partial h}{\partial t} + \lambda^T (f - \dot{\mathbf{x}}) - \dot{\mathbf{x}} \left(\frac{\partial h}{\partial t} - \lambda \right) \right]_{t=t_f} = \left(g + \lambda^T f + \frac{\partial h}{\partial t} \right)_{t=t_f} = 0$$

The $2n + 1$ relationship is:

$$H(\mathbf{x}(t_f), \mathbf{u}(t_f), \lambda(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}(t_f), t_f) = 0$$

Case II Final state free. The two boundary conditions are derived from the two transversality conditions:

$$\lambda(t_f) = \frac{\partial h}{\partial \mathbf{x}}(\mathbf{x}(t_f), t_f)$$

$$H(\mathbf{x}(t_f), \mathbf{u}(t_f), \lambda(t_f), t_f) + \frac{\partial h}{\partial t}(\mathbf{x}(t_f), t_f) = 0$$

Example 2.1.1 *The system*

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t) \end{aligned}$$

is to be controlled so that its control effort is conserved; that is the performance measure

$$J(u) = \int_{t_0}^{t_f} \frac{1}{2} u^2(t) dt$$

is to be minimized. The admissible states and controls are not bounded. Find necessary conditions that must be satisfied for optimal control.

The first step is to form the hamiltonian

$$H(\mathbf{x}, u, \lambda) = \frac{1}{2} u^2(t) + \lambda_1(t) x_2(t) + \lambda_2(t) (-x_2(t) + u(t))$$

From equations (2.5) and (2.6) the necessary conditions for optimality are:

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) + \lambda_2(t) \end{aligned} \tag{2.8}$$

and

$$0 = \frac{\partial H}{\partial u} = u(t) + \lambda_2(t) \quad (2.9)$$

If (2.9) is solved for $u^*(t)$ and substituted into the state equations, we have

$$u^*(t) = -\lambda_2(t)$$

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) - \lambda_2(t) \end{aligned} \quad (2.10)$$

Equations (2.10) and (2.8) are a set of 4 first-order constant-coefficient differential equations. From (2.8):

$$\lambda_1(t) = c_1; \quad \dot{\lambda}_2(t) = -c_1 + \lambda_2(t) \Rightarrow \lambda_2(t) = c_2 e^t + c_1$$

$$u^*(t) = -c_2 e^t - c_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_2 - \lambda_2 = -x_2 - c_2 e^t - c_1; \Rightarrow x_2 = c_3 e^{-t} + c_4 e^t;$$

The constant c_4 from the particular solution of x_2 is calculated by replacing the general solution into the equation

$$-c_3 e^{-t} + c_4 e^t + c_3 e^{-t} + c_4 e^t = -c_2 e^t - c_1 \Rightarrow c_4 = -\frac{c_2}{2} - \frac{c_1}{2} e^{-t}$$

The general solution of the state equation result:

$$x_2(t) = c_3 e^{-t} - \frac{c_2}{2} e^t - \frac{c_1}{2}$$

$$x_1(t) = -c_3 e^{-t} - \frac{c_2}{2} e^t - \frac{c_1}{2} t + c_4$$

Suppose the initial and final states are specified:

$$\mathbf{x}(0) = [0 \ 0]^T; \quad \mathbf{x}(2) = [5 \ 2]^T$$

We calculate the constants from:

$$x_1(0) = -c_3 - \frac{c_2}{2} + c_4 = 0$$

$$x_2(0) = c_3 - \frac{c_2}{2} - \frac{c_1}{2} = 0$$

$$x_1(2) = -c_3 e^{-2} - \frac{c_2}{2} e^2 - c_1 + c_4 = 5$$

$$x_2(2) = c_3 e^{-2} - \frac{c_2}{2} e^2 - \frac{c_1}{2}$$

Example 2.1.2 *For the system*

$$\begin{aligned}\dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= -x_2(t) + u(t)\end{aligned}$$

find the necessary conditions for an optimal control $u^*(t)$ that minimizes the performance measure:

$$J(u) = \frac{1}{2}[x_1(2) - 5]^2 + \frac{1}{2}[x_2(2) - 2]^2 + \frac{1}{2} \int_0^2 u^2(t) dt$$

The initial conditions are $\mathbf{x}(0) = 0$, and $\mathbf{x}(2)$ are unspecified. The hamiltonian is:

$$H(\mathbf{x}, u, \lambda) = \frac{1}{2}u^2(t) + \lambda_1(t)x_2(t) + \lambda_2(t)(-x_2(t) + u(t))$$

$$\frac{\partial H}{\partial u} = u + \lambda_2 = 0 \Rightarrow u^*(t) = -\lambda_2(t)$$

The costate equations:

$$\begin{aligned}\dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2} = -\lambda_1(t) + \lambda_2(t)\end{aligned}$$

have the final conditions calculated from (2.7):

$$\begin{aligned}\lambda_1(t_f) &= \frac{\partial h}{\partial x_1(2)}|_{t=t_f=2} = x_1(2) - 5 = \lambda_1(2) \\ \lambda_2(t_f) &= \frac{\partial h}{\partial x_2(2)}|_{t=t_f=2} = x_2(2) - 2 = \lambda_2(2)\end{aligned}$$

Comparing to the previous example, the modified performance measure affects only the boundary conditions at $t = 2$. The general results obtained in the previous example are valid here too. Only the constants will result different.

Bibliography

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